

Wavy way to the Kerr metric and the quantum nature of its ring singularity

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From inherent non-linearity two gravitational waves, unless they are unidirectional, fail to satisfy a superposition law. They collide to develop a new spacetime carrying the imprints of the incoming waves. Same behaviour is valid also for any massless lightlike field. As a result of the violent collision process either a naked singularity or a Cauchy horizon (CH) develops. It was shown by Chandrasekhar and Xanthopoulos (CX) that a particular class of colliding gravitational waves (CGW) spacetime is locally isometric to the Kerr metric for rotating black holes. This relation came to be known as the CX duality. Such a duality can be exploited as an alternative derivation for the Kerr metric as we do herein. Not each case gives rise to a CH but those which do are transient to a black hole state provided stability requirements are met. These classical considerations can be borrowed to shed light on black hole formation in high energy collisions. Their questionable stability and many other sophisticated agenda, we admit that await for a full - fledged quantum gravity. Yet, to add an element of novelty, a quantum probe is sent in the plane $\theta = \pi/2$ to the naked ring singularity of Kerr which develops for the overspinning case ($a > M$) to test it from a quantum picture. We show that the spatial operator of the reduced Klein-Gordon equation has a unique self-adjoint extension. As a result, the classical Kerr's ring singularity is healed and becomes quantum regular. Our poetic message of the paper is summarized as

*Let there be light
that collide with might
to disperse the night
and create holes that are white*

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I. INTRODUCTION

No doubt the intellectual capacity and motivation boost added to Einstein's general relativity (GR) and astrophysics/cosmology by the Kerr metric [1] has been enormous. It is the rotational extension of the famous Schwarzschild solution [2] discovered long ago as early as 1916. As the Schwarzschild metric represents a static black hole solution of Einstein's equations, the Kerr metric represents its spinning version. Mass and angular momentum are the physical parameters that characterize the Kerr black hole. Its original discovery in a pre-computer era such as early 1960's by R. P. Kerr with the only available logistics of paper, pencil and of course certain level of intuition and inspiration was almost a miracle. And since the year 2015 marks the centennial of GR, it should appropriately be in order to add some minor contributions as an alternative derivation of the Kerr solution. There are already interesting review articles about the Kerr metric in the literature [3–5, 24]. Our route to Kerr will be entirely different from our predecessors, it will be a "wavy way" since we incorporate the collision of gravitational waves.

Being a highly nonlinear theory, rotational effects in GR can not be encompassed by perturbation techniques. That is, to small order (weak) of rotation, the metric written down does not reflect the complete properties of an exact solution. Recall simply that in a non-linear theory, superposition of two solutions is not a solution. In particular, fast rotation that highly distorts the spacetime around leads to acausal connections, closed time-like curves and probably spacetime wormholes. To say the least, the subject of singularities alone takes an entire life time for an exhaustive study. For these and many other reasons, Kerr metric made a mark in the history of science, in particular of relativity and naturally deserves further comments.

In this modern era of undergoing collision experiments at the Large Hadron Collider (LHC) at CERN, it is our aim to draw attention that the Kerr metric emerges locally as a result of colliding gravitational waves (CGW). These incoming waves consist of an impulsive and shock components in superposition. This result has been known for at least three decades by the researchers on colliding waves in GR, however, to inform the rest of the physics community

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as well makes one of the principal aim of the present article.

Collision of gravitational waves with collinear polarization were discovered first by Khan and Penrose (KP) [6] and Szekeres [7, 8]. The difference between colliding waves and stationary axially symmetric spacetimes is that the former has two spacelike while the latter has one spacelike and one timelike Killing vectors. This implies automatically that the spacetime of colliding waves is a time dependent, dynamic spacetime. Extension of the KP solution to non-collinear polarization was achieved by Nutku and Halil (NH) [9]. The relative polarization angle created a cross - term in the metric with a naked spacetime singularity weaker than that of KP solution. By changing the profile of the incoming waves, new solutions were generated which are listed in the book by Griffiths [10]. The most interesting among those that makes at the same time the subject matter of the present article is the solution found by Chandrasekhar and Xanthopoulos (CX) [11], (about their formalism see also the work by Chandrasekhar and Ferrari (CF)[12]). They employed the adjoint solution of the Ernst equation [13] rather than the standard solution used in the derivation of the NH solution for colliding waves. The result was remarkable in the sense that the emerging metric in the interaction region was locally isometric/transformable to the Kerr metric. Vanishing of the relative polarization angle between the incoming waves transforms the resulting metric automatically to the Schwarzschild metric, as expected. Being locally isometric to the Kerr metric, the CX metric inside the interaction region of colliding waves is also type-D in Petrov classification, so that the geodesics and Hamilton-Jacobi equations admit solutions by separation of variables. On the other hand, the KP/NH geodesics are not integrable by separation of variables. These are in fact of Petrov type-I which lacks the separability conditions [14]. Let us add also that from the universality of the gravitational interaction collision of any other gauge fields such as electromagnetism or more generally the non-abelian Yang-Mills fields yield analogous results to gravitational fields coupled with physical sources. Thus, assuming that the Kerr metric was not discovered before, it could be discovered, as a result of two CGW accompanied with a coordinate transformation.

Is there anything deeper that we hover around in all this endeavor ?. If black holes are considered as matter with their miniscale forms as particles should there not be a corresponding wave dual to it ?; the spirit of wave - particle duality. After all, the wave - particle duality lies at the heart of physics. Based on this analogy, the adjoint solution of CX has both particle - like and wave - like aspects whereas the normal colliding wave spacetime (the NH solution) has purely wave - like aspects. Let us add that Penrose's remark [15], that every (material) spacetime admits a plane wave spacetime as a limit is not also independent from this reality. One step further, in the dynamical Near - Horizon - Geometry, we identify instead of the plane wave of Penrose the colliding plane waves [16].

It is shown in the CX duality / isometry that the relative polarization angle of the waves taking part in the collision becomes proportional to the angular momentum of the Kerr metric. In other words, the relative polarization angle of the colliding waves transforms in the isometry to the rotational degree of the resulting metric. The "equivalent mass" degree of freedom in the local isometry is derived from the curvature of spacetime. The lesson to extract from such a result is that in the ultra high energy collision experiments undergoing at CERN we may search for micro Kerr black holes [17]. Adding to this the interesting Banados - Silk - West (BSW) [18] effect that the center - of - mass (CM) energy in the particle collisions in the vicinity of the event horizon of the created micro black holes grow unbounded makes the high energy collision experiments further important. A severe handicap must be supplemented, however, that the event horizon of such a black hole is not stable against the slightest perturbation and decays instantly [19]. At the quantum level this result can not be detached from the classical picture which says that the Cauchy horizon developed in colliding waves can not be stable [20-22].

We undertake also to investigate from quantum approach the reality of timelike naked singularities when the rotation parameter dominates the mass parameter. Unlike the formation of naked singularities in static spacetimes, naked singularity in the Kerr metric has an exceptional feature. In static black hole solutions, the naked singularity is central and located at $r = 0$. Hence, all the trajectories hit the singularity once they cross the event horizon. On the other hand, the naked singularity in the Kerr black hole is encountered when $r = 0$ and $\theta = \pi/2$. The surface $r = 0$ is a disc and the timelike naked ring singularity forms the boundary of this disc inside the toroidal ergo region. However, the trajectories approaching this ring singularity when $\theta \neq \pi/2$, automatically miss the singularity, to encounter regular points. In fact, the scenario for overspinning case (overextremal, $a > M$) has been questioned by several authors [40-42], for the sake of weak cosmic censorship. The analysis in these studies have revealed that by capturing particles, an initial extremal ($a = M$) black hole cannot decay into an overspinning case. As a consequence, the possibility of having an overspinning black hole is unlikely to occur, and excludes the possibility to have a naked singularity, at least for a restricted class of scenario. As it was stated in [42], more detailed studies that cover all possible scenarios are needed for the resolution of the overspinning problem.

In this article, the overspinning case of the Kerr's solution that admits timelike naked singularity is investigated with quantum particles obeying the Klein-Gordon equation. The Horowitz-Marolf (HM) [30], criterion developed for static spacetimes is extended to stationary spacetimes for a specific wave mode in which the temporal and spatial parts of the Klein-Gordon equation are separated. Thus, instead of point particle probe, the singularity is probed with waves for exploring its quantum nature. Our finding shows that the ring singularity of Kerr becomes quantum

regular.

In view of all these one may conclude that there exist structural similarities in the mathematical theories between the two seemingly unrelated topics of GR, namely, the CGW and black holes. This structural similarity has become more apparent with the discovery of the Ernst formalism. This is the new formulation of the stationary axially symmetric gravitational field problem formulated by Ernst in 1968 [13]. Ernst formalism which leads to the Ernst equation is a cornerstone in obtaining an exact analytic solution to the field equations of GR. Its symmetries are described in detail in the book "The Mathematical Theory of Black Holes" by Chandrasekhar [23]. The formulation of CGW and black holes admit the same type of Ernst equation. For the sake of completeness, we wish to review the formalism of Ernst in section II, for the comprehensive analysis which play a key role in the mathematical structural similarity in theories of black holes and of CGW. Section III is devoted for exploring this similarity. First, the derivation of the stationary axisymmetric black hole solution is briefly explained in Ernst formalism. In the preceding subsection, the problem of CGW which becomes isometric to the Kerr black hole is explained in detail. In section IV, the quantum nature of the Kerr's ring singularity for the overspinning case is investigated within the framework of quantum mechanics. The paper ends with a conclusion and discussion in section V.

II. THE ERNST FORMALISM

The Ernst formalism introduces an alternative derivation of the field equations for a uniformly rotating axially symmetric source. This new formalism involves the use of complex function ξ which is independent of the azimuthal coordinate. Once ξ is found, a corresponding axially symmetric solutions of Einstein's or Einstein - Maxwell equations are constructed. In the present study, our focus will be on the derivation of the Kerr metric and hence, we shall consider the vacuum Einstein's field equations.

Our starting point is to adopt the metric known as the Papapetrou - Weyl form for rotating axially symmetric fields given by

$$ds^2 = f^{-1} [e^{2\gamma} (dz^2 + d\rho^2) + \rho^2 d\varphi^2] - f (dt - \omega d\varphi)^2, \quad (1)$$

in which $f = f(\rho, z)$, $\omega = \omega(\rho, z)$ and $\gamma = \gamma(\rho, z)$. Since the metric functions are independent of time coordinate, it is called stationary axially symmetric. If $\omega = 0$, the metric becomes static axially symmetric. As a requirement of the formalism, the following Lagrangian density is introduced which involves only the metric functions f and ω ,

$$\mathcal{L} = -\frac{1}{2}\rho f^{-2} \vec{\nabla} f \cdot \vec{\nabla} f + \frac{1}{2}\rho^{-1} f^2 \vec{\nabla} \omega \cdot \vec{\nabla} \omega. \quad (2)$$

By applying the method of variation with respect to f and ω , the following field equations are obtained

$$f \nabla^2 f = \vec{\nabla} f \cdot \vec{\nabla} f - \rho^{-2} f^4 \vec{\nabla} \omega \cdot \vec{\nabla} \omega, \quad (3)$$

$$\vec{\nabla} \cdot (\rho^{-2} f^2 \vec{\nabla} \omega) = 0. \quad (4)$$

Let us add that the mathematical operators gradient ($\vec{\nabla}$), divergence ($\vec{\nabla} \cdot$) and Laplacian (∇^2) are all defined on the flat spacetime,

$$ds_0^2 = d\rho^2 + dz^2 + \rho^2 d\varphi^2. \quad (5)$$

Note also that if $\omega = 0$, Eq.(3) becomes $f \nabla^2 f = (\nabla f)^2$ and can be integrated easily to find the solutions. The obtained solution for this particular case is known as Weyl solution.

It is well - known from the vector calculus that for any vector field \vec{A} , we have,

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0. \quad (6)$$

If we compare Eq.(6) with that of Eq.(4), we have

$$\rho^{-2} f^2 \vec{\nabla} \omega = \vec{\nabla} \times \vec{A}. \quad (7)$$

Since the considered spacetime is axially symmetric, it implies independent of φ dependence and as a result,

$$\omega_{,\varphi} = 0. \quad (8)$$

The above equation imposes the condition that the \hat{e}_φ component of the right hand side of the Eq.(7) is zero. Hence, the Eq.(7) yields,

$$\omega_{,\rho} = \rho f^{-2} (\rho A_{\varphi,z} - A_{z,\varphi}), \quad (9)$$

$$\omega_{,z} = \rho f^{-2} (A_{\rho,\varphi} - \rho A_{\varphi,\rho} - A_\varphi). \quad (10)$$

At this stage we define a new function F such that

$$F_{,\rho} = A_\rho, \quad \text{and} \quad F_{,z} = A_z, \quad (11)$$

and by introducing a new function Φ as

$$\Phi = F_{,\varphi} - \rho A_\varphi, \quad (12)$$

equations (9) and (10) can be written in terms of the new function Φ as,

$$\omega_{,\rho} = -\rho f^{-2} \Phi_{,z} \quad (13)$$

$$\omega_{,z} = \rho f^{-2} \Phi_{,\rho}. \quad (14)$$

Solution to ω from this pair should also satisfy the integrability condition

$$\omega_{,\rho z} = \omega_{,z\rho}. \quad (15)$$

This integrability condition implies that the new function Φ should also satisfy the same integrability condition

$$\Phi_{,\rho z} = \Phi_{,z\rho}. \quad (16)$$

As a consequence, Eqs.(3) and (4) can be written in terms of new function Φ as,

$$f \nabla^2 f = \left(\vec{\nabla} f \right)^2 - \left(\vec{\nabla} \Phi \right)^2, \quad (17)$$

$$\vec{\nabla} \cdot \left(f^{-2} \vec{\nabla} \Phi \right) = 0. \quad (18)$$

Once equations (17) and (18) are solved, the other metric functions γ and ω can be obtained by integration. The equations (17) and (18) can be combined if a new function ξ is defined as

$$\xi = f + i\Phi, \quad (19)$$

which yields,

$$(\text{Re}\xi) \nabla^2 \xi = (\nabla \xi)^2. \quad (20)$$

This equation is called the Ernst equation for vacuum. Another alternative way of writing the Ernst equation is to define another function as,

$$\xi = \frac{E - 1}{E + 1}, \quad (21)$$

leading to,

$$(EE^* - 1) \nabla^2 E = 2E^* (\nabla \xi)^2, \quad (22)$$

in which "*" denotes the complex conjugation. It should be supplemented once more that the operators are to be evaluated on a base manifold (5) with φ a Killing coordinate.

Let us note that up to this point we have introduced the Ernst formalism based on the the Weyl - Papapetrou metric and coordinates ρ and z . However, solutions in these coordinates are not much tractable and this enforces us to consider alternative coordinates. One such system is the prolate coordinates x, y related to ρ and z by

$$\begin{aligned} \rho &= \sqrt{x^2 - 1} \sqrt{1 - y^2}, \\ z &= xy. \end{aligned} \quad (23)$$

By transforming all operators and equations in to the (x, y) coordinates the Kerr solution follows as the simplest complex solution to the Ernst equation. We have

$$E = px - i q y \quad (24)$$

with constants p and q satisfying $p^2 + q^2 = 1$. In terms of the latter coordinates Kerr's metric reads

$$\begin{aligned} ds^2 = & \frac{(p^2 x^2 + q^2 y^2 - 1)}{(px + 1)^2 + q^2 y^2} \left[dt - \frac{2q(1 - y^2)(px + 1)}{p^2 x^2 + q^2 y^2 - 1} d\varphi \right]^2 \\ & - \frac{(px + 1)^2 + q^2 y^2}{p^2} \left[\frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right] \\ & - \frac{(px + 1)^2 + q^2 y^2}{p^2 x^2 + q^2 y^2 - 1} (x^2 - 1)(1 - y^2) d\varphi^2. \end{aligned} \quad (25)$$

This form of the Kerr metric is a convenient form to be related with the colliding wave metric that will be introduced in the next section. From this form of the Kerr metric by a simple transformation we can transform it into the Boyer - Lindquist form which is the most familiar form of the metric: we employ the transformation

$$px + 1 = \frac{r}{M}, \quad t = t, \quad qy = \frac{a}{M} \cos \theta, \quad \varphi = \varphi, \quad p = \frac{\sqrt{M^2 - a^2}}{M}, \quad q = \frac{a}{M}. \quad (26)$$

and obtain

$$\begin{aligned} ds^2 = & dt^2 - (r^2 + a^2 \cos^2 \theta) \left[d\theta^2 + \frac{dr^2}{r^2 + a^2 - 2Mr} \right] \\ & - (r^2 + a^2) \sin^2 \theta d\varphi^2 - \frac{2Mr}{r^2 + a^2 \cos^2 \theta} (dt - a \sin^2 \theta d\varphi)^2 \end{aligned} \quad (27)$$

It can be readily seen that for $a = 0$ this reduces to the standard Schwarzschild line element.

III. STRUCTURAL SIMILARITY OF THE MATHEMATICAL THEORY OF BLACK HOLES AND CGW.

Spacetimes admitting two Killing fields provide the necessary background for both the theory of black holes and the theory of CGW. Since we are interested with stationary axially symmetric metrics; the metric functions representing black holes are independent of the time t and of the azimuthal angle φ . On the other hand, the metric functions representing colliding gravitational waves are independent of two spacelike coordinates (x^1, x^2) ranging from $-\infty$ to $+\infty$, metric functions depend on the time t and the spacelike coordinate x^3 which is normal to the (x^1, x^2) - planes.

A. The Conjugate Solution of the Stationary Axisymmetric Black Hole Spacetime

The metric describing the stationary axisymmetric black holes can also be written in the form adopted by Chandrasekhar [23],

$$ds^2 = \sqrt{\Delta \delta} \left[\chi dt^2 - \frac{1}{\chi} (d\varphi - \omega dt)^2 \right] - e^{\mu_2 + \mu_3} \sqrt{\Delta} \left[\left(\frac{d\eta}{\Delta} \right)^2 + \left(\frac{d\mu}{\delta} \right)^2 \right], \quad (28)$$

in which

$$\Delta = \eta^2 - 1, \quad \delta = 1 - \mu^2, \quad (\mu = \cos \theta), \quad (29)$$

where η is the radial coordinate, χ, ω and $\mu_2 + \mu_3$ are the metric functions. The metric function ω represents the rotation parameter. As we have experienced from the Ernst formalism, the whole idea is to solve for the metric functions χ and ω . The remaining metric function $\mu_2 + \mu_3$, can be obtained by simple integral.

The conjugate metric of the metric (28) can be obtained by applying the following transformations;

$$t \rightarrow +i\varphi \text{ and } \varphi \rightarrow -it. \quad (30)$$

This conjugation modifies the metric (28) in the following form

$$ds^2 = \sqrt{\Delta\delta} \left[\tilde{\chi} dt^2 - \frac{1}{\tilde{\chi}} (d\varphi - \tilde{\omega} dt)^2 \right] - e^{\mu_2 + \mu_3} \sqrt{\Delta} \left[\left(\frac{d\eta}{\Delta} \right)^2 + \left(\frac{d\mu}{\delta} \right)^2 \right], \quad (31)$$

in which,

$$\tilde{\chi} = \frac{\chi}{\chi^2 - \omega^2}, \quad \tilde{\omega} = \frac{\omega}{\chi^2 - \omega^2} \quad (32)$$

Consequently, if the pair of (χ, ω) is a solution to the field equations, then the new pair $(\tilde{\chi}, \tilde{\omega})$ is also a solution to the same field equations. As a requirement of the Ernst formalism, in place of χ and ω , we define two new functions Ψ and Φ such that;

$$\Psi = \frac{\sqrt{\Delta\delta}}{\chi}, \quad (33)$$

and Φ is regarded as the potential for ω and defined by

$$\Phi_{,\eta} = \frac{\delta}{\chi^2} \omega_{,\mu}, \quad \text{and} \quad \Phi_{,\mu} = -\frac{\Delta}{\chi^2} \omega_{,\eta}. \quad (34)$$

One can also write these equations (33) and (34) in terms $\tilde{\chi}$ and $\tilde{\omega}$ as;

$$\tilde{\Psi} = \frac{\sqrt{\Delta\delta}}{\tilde{\chi}}, \quad (35)$$

$$\tilde{\Phi}_{,\eta} = \frac{\delta}{\tilde{\chi}^2} \tilde{\omega}_{,\mu}, \quad \text{and} \quad \tilde{\Phi}_{,\mu} = -\frac{\Delta}{\tilde{\chi}^2} \tilde{\omega}_{,\eta}. \quad (36)$$

The functions Ψ, Φ and $\tilde{\Psi}, \tilde{\Phi}$ can be combined into the pairs of complex functions as

$$Z^\dagger = \Psi + i\Phi, \quad \text{and} \quad \tilde{Z}^\dagger = \tilde{\Psi} + i\tilde{\Phi}, \quad (37)$$

which is followed by an introduction of new function E as,

$$E^\dagger = \frac{Z^\dagger - 1}{Z^\dagger + 1}, \quad \text{and} \quad \tilde{E}^\dagger = \frac{\tilde{Z}^\dagger - 1}{\tilde{Z}^\dagger + 1}. \quad (38)$$

These functions admits the following Ernst equation,

$$(1 - |E|^2) \{ [\Delta E_{,\eta}]_{,\eta} - [\delta E_{,\mu}]_{,\mu} \} = -2E^* \left[\Delta (E_{,\eta})^2 - \delta (E_{,\mu})^2 \right]. \quad (39)$$

in which E can be replaced by E^\dagger or \tilde{E}^\dagger .

The same solution for the Ernst equation of the last section namely $E = p\eta - iq\mu$, can be employed here in which the prolate coordinates x and y are changed into η and μ . The relation of these coordinates to the null coordinates u and v will be described in the next section.

B. Formulation of the CGW Problem in Double Null Coordinates

The straightforward technique for CGW is to devise the whole spacetime into four regions and formulate it as an initial-value problem, as depicted in figure 1. Usually, Region I, represents the flat Minkowski space in which no gravitational waves are present. Note that CX interchanges regions I and IV in their papers. Regions II and III, are the plane symmetric regions which contains gravitational plane waves that participates in the collision. Region IV is the interaction region which describes the nonlinear interaction of gravitational waves in which the resulting spacetime is no more plane symmetric. The formulation for initial-value problem requires first to define the waves in the incoming regions, and solve the necessary field equations in the region of interaction.

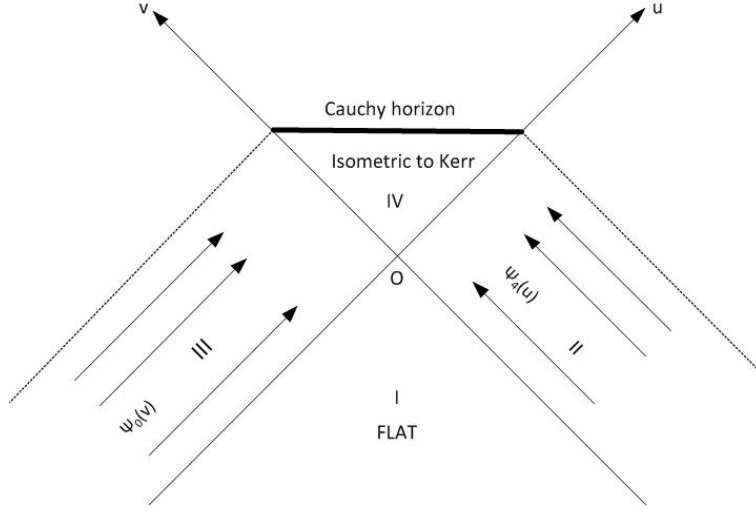


FIG. 1: The figure depicts collision of two incoming gravitational plane waves $\Psi_4(u)$ (from Region II) and $\Psi_0(v)$ (from Region III). The collision takes place at the origin O ($u = 0$ and $v = 0$). The post - collision region (Interaction region or Region IV) is isometric to the Kerr spacetime. In the diagram, coordinates x and y are orthogonal to the (u, v) plane so that they are suppressed. The hypersurface $u^2 + v^2 = 1$, locates a Cauchy horizon (CH), not a spacetime singularity, beyond which the spacetime can be extended analytically to new spacetimes. It should be added that the stability of the CH against various perturbations remains as an unsettled dispute. Researchers so far infer about an unstable CH.

However, an alternative method is developed by CF which uses the Ernst formalism for the formulation of the problem of CGW. In this method, first the solution in the region of interaction is found and then by making an extension the waves that participate in the collision is found. This is the method that we shall employ to drive the Kerr metric in this article.

The adopted line element for the description of the CGW is the Szekeres line element given by,

$$ds^2 = 2e^{-M}dudv - e^{-U} \{ (e^V dx^2 + e^{-V} dy^2) \cosh W - 2 \sinh W dx dy \}, \quad (40)$$

in which $M = M(u, v)$, $U = U(u, v)$, $V = V(u, v)$ and $W = W(u, v)$ are the metric functions to be found, all depends on the null coordinates u and v in the region of interaction. This metric admits two commuting spacelike Killing vectors $\xi_1 = \partial_x$ and $\xi_2 = \partial_y$. Note that in Chandrasekhar's notation $x = x^1$ and $y = x^2$. The vacuum Einstein equations governing the solution to the metric functions are

$$U_{uv} = U_u U_v, \quad (41)$$

$$2U_{vv} = U_v^2 + W_v^2 + V_v^2 \cosh^2 W - 2U_v M_v, \quad (42)$$

$$2U_{uu} = U_u^2 + W_u^2 + V_u^2 \cosh^2 W - 2U_u M_u, \quad (43)$$

$$2V_{uv} = U_u V_v + U_v V_u - 2(V_u W_v + V_v W_u) \tanh W, \quad (44)$$

$$2M_{uv} = -U_u V_v + W_v W_u + V_u V_v \cosh^2 W, \quad (45)$$

$$2W_{uv} = U_u W_v + U_v W_u + 2V_u V_v \sinh W \cosh W. \quad (46)$$

Note that these equations, with the exceptions of (42) and (43) follow from a variational principle of the Lagrangian,

$$L = e^{-U} \{ M_u U_v + M_v U_u + U_u U_v - V_u V_v \cosh^2 W - W_v W_u \}, \quad (47)$$

where our notation is such that a subscript u / v letter implies partial derivative. The equations (42) and (43) are simply the integrability conditions for the other equations. The set of the above field equations can be solved by employing the Ernst formalism. In doing this, the following complex valued function is defined,

$$Z = \chi + iq_2, \quad (48)$$

where

$$\chi = \frac{e^{-V}}{\cosh W}, \quad \text{and} \quad q_2 = e^{-V} \tanh W, \quad (49)$$

such that the line element (40) becomes

$$ds^2 = 2e^{-M} dudv - e^{-U} \left[\chi dy^2 + \frac{1}{\chi} (dx - q_2 dy)^2 \right]. \quad (50)$$

The field equation (41) can be integrated to give,

$$e^{-U} = f(u) + g(v), \quad (51)$$

in which $f(u)$ and $g(v)$ are arbitrary functions of their arguments. These functions can be chosen so that it satisfies the required boundary conditions on the null boundaries $u = 0$, $v = 0$ and hence given by,

$$e^{-U} = 1 - u^2 - v^2. \quad (52)$$

Note that within the context of colliding waves the coordinates u and v are to be considered with the Heaviside unit step functions $\theta(u)$ and $\theta(v)$ where

$$\theta(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}. \quad (53)$$

That is, we must have the substitutions

$$u \rightarrow u\theta(u), \quad \text{and} \quad v \rightarrow v\theta(v).$$

Obviously the choice,

$$e^{-U} = 1 - u\theta(u) - v\theta(v),$$

will give rise to Dirac delta functions in the second derivatives, so this must be excluded as a possible solution to Eq.(41).

By defining a new set of coordinates [9],

$$\eta = u\sqrt{1-v^2} + v\sqrt{1-u^2}, \quad \text{and} \quad \mu = u\sqrt{1-v^2} - v\sqrt{1-u^2}, \quad (54)$$

the metric that describes the collision of gravitational waves in the region of interaction is transformed to the following form,

$$ds^2 = e^{\nu+\mu_3} \sqrt{\Delta} \left[\frac{(d\eta)^2}{\Delta} - \frac{(d\mu)^2}{\delta} \right] - \sqrt{\Delta\delta} \left[\chi (dy)^2 + \frac{1}{\chi} (dx - q_2 dy)^2 \right], \quad (55)$$

where η defines the time from the instant of the collision, μ defines the distance in the normal direction to the spacelike (x, y) - planes with,

$$\Delta = 1 - \eta^2, \quad \text{and} \quad \delta = 1 - \mu^2, \quad (56)$$

and χ , $\nu + \mu_3$ and q_2 are the metric functions to be found. It should be noted that the metric functions q_2 , measures the variation in the polarization of the gravitational waves. If $q_2 = 0$, the gravitational waves are said to be linearly polarized. Note also that in the transformation $u \rightarrow \sin u$ and $v \rightarrow \sin v$, instead of (54) is the choice employed by CX [12]. In accordance with the latter choice we have $\eta = \sin(u + v)$ and $\mu = \sin(u - v)$ with possible scalings of the null coordinates, such as $u \rightarrow au$ and $v \rightarrow bv$ with (a, b) constants.

Following the similar steps as was used in the mathematical theory of black holes [23], the new parametrization;

$$\chi + iq_2 = Z = \frac{1 + E}{1 - E}, \quad (57)$$

one obtains the Ernst equation in terms of Z as

$$(Z + Z^*) \left[(\Delta Z_{,\eta})_{,\eta} - (\delta Z_{,\mu})_{,\mu} \right] = 2 \left[\Delta (Z_{,\eta})^2 - \delta (Z_{,\mu})^2 \right], \quad (58)$$

and in terms of E as,

$$(1 - |E|^2) \{ [\Delta E_{,\eta}]_{,\eta} - [\delta E_{,\mu}]_{,\mu} \} = -2E^* \left[\Delta (E_{,\eta})^2 - \delta (E_{,\mu})^2 \right], \quad (59)$$

which is the same equation of (39). The equations related to metric function $\nu + \mu_3$, can be written in terms of E and given by

$$\frac{\mu}{\delta} (\nu + \mu_3)_{,\eta} + \frac{\eta}{\Delta} (\nu + \mu_3)_{,\mu} = -\frac{1}{\chi^2} (\chi_{,\eta} \chi_{,\mu} + q_{2,\eta} q_{2,\mu}) = -2 \frac{E_{,\eta} E_{,\mu}^* + E_{,\eta}^* E_{,\mu}}{(1 - |E|^2)^2}, \quad (60)$$

and

$$\begin{aligned} 2\eta (\nu + \mu_3)_{,\eta} + 2\mu (\nu + \mu_3)_{,\mu} &= \frac{3}{\Delta} + \frac{1}{\delta} - \frac{1}{\chi^2} \left\{ \Delta [(\chi_{,\eta})^2 + (q_{2,\eta})^2] + \delta [(\chi_{,\mu})^2 + (q_{2,\mu})^2] \right\} \\ &= \frac{3}{\Delta} + \frac{1}{\delta} - \frac{4}{(1 - |E|^2)^2} \left[\Delta |E_{,\eta}|^2 + \delta |E_{,\mu}|^2 \right]. \end{aligned} \quad (61)$$

The derivation of the metric function q_2 , is possible from a potential Φ , as in the case for the metric function ω of the black hole case, we have

$$\Phi_{,\eta} = \frac{\delta}{\chi^2} q_{2,\mu} \quad \text{and} \quad \Phi_{,\mu} = \frac{\Delta}{\chi^2} q_{2,\eta}, \quad (62)$$

and defining

$$Z^\dagger = \Psi + i\Phi = \frac{1 + E^\dagger}{1 - E^\dagger}, \quad (63)$$

in which

$$\Psi = \frac{\sqrt{\Delta\delta}}{\chi}. \quad (64)$$

With this formalism, the corresponding equation for Z^\dagger and the equation for E^\dagger , which is the Ernst equation are obtained respectively,

$$(Z^\dagger + (Z^\dagger)^*) \left[(\Delta (Z^\dagger)_{,\eta}^*)_{,\eta} - (\delta (Z^\dagger)_{,\mu}^*)_{,\mu} \right] = 2 \left[\Delta ((Z^\dagger)_{,\eta}^*)^2 - \delta ((Z^\dagger)_{,\mu}^*)^2 \right], \quad (65)$$

$$(1 - |(E^\dagger)|^2) \left\{ [\Delta (E^\dagger)_{,\eta}]_{,\eta} - [\delta (E^\dagger)_{,\mu}]_{,\mu} \right\} = -2 (E^\dagger)^* \left[\Delta ((E^\dagger)_{,\eta})^2 - \delta ((E^\dagger)_{,\mu})^2 \right], \quad (66)$$

and the equations governing $\nu + \mu_3$, can be expressed in terms of E^\dagger as

$$\begin{aligned} \frac{1}{\chi^2} (\chi_{,\eta} \chi_{,\mu} + q_{2,\eta} q_{2,\mu}) &= (\ln \chi)_{,\eta} (\ln \chi)_{,\mu} + \frac{\Phi_{,\eta} \Phi_{,\mu}}{\Psi^2} \\ &= \left(\frac{\eta}{\Delta} + \frac{\Psi_{,\eta}}{\Psi} \right) \left(\frac{\mu}{\delta} + \frac{\Psi_{,\mu}}{\Psi} \right) + \frac{\Phi_{,\eta} \Phi_{,\mu}}{\Psi^2} \\ &= \frac{\mu}{\delta} \left[\ln \frac{\Psi}{\sqrt[4]{\Delta\delta}} \right]_{,\eta} + \frac{\eta}{\Delta} \left[\ln \frac{\Psi}{\sqrt[4]{\Delta\delta}} \right]_{,\mu} + \frac{\Phi_{,\eta} \Phi_{,\mu} + \Psi_{,\eta} \Psi_{,\mu}}{\Psi^2}. \end{aligned} \quad (67)$$

Hence, equation (60) transforms to the form

$$\begin{aligned} \frac{\mu}{\delta} \left[(\nu + \mu_3) + \ln \frac{\Psi}{\sqrt[4]{\Delta\delta}} \right]_{,\eta} + \frac{\eta}{\Delta} \left[(\nu + \mu_3) + \ln \frac{\Psi}{\sqrt[4]{\Delta\delta}} \right]_{,\mu} \\ = -\frac{\Phi_{,\eta} \Phi_{,\mu} + \Psi_{,\eta} \Psi_{,\mu}}{\Psi^2} = -2 \frac{E_{,\eta}^\dagger (E^\dagger)^*_{,\mu} + (E^\dagger)^*_{,\eta} E_{,\mu}^\dagger}{(1 - |E^\dagger|^2)^2}, \end{aligned} \quad (68)$$

and the equation (61) transforms to the form

$$2\eta \left[(\nu + \mu_3) + \ln \frac{\Psi}{\sqrt[4]{\Delta\delta}} \right]_{,\eta} + 2\mu \left[(\nu + \mu_3) + \ln \frac{\Psi}{\sqrt[4]{\Delta\delta}} \right]_{,\mu} = \frac{3}{\Delta} + \frac{1}{\delta} - \frac{4}{(1 - |E^\dagger|^2)^2} \left[\Delta |E^\dagger_{,\eta}|^2 + \delta |E^\dagger_{,\mu}|^2 \right]. \quad (69)$$

1. Chandrasekhar - Xanthopoulos (CX) Solution

The fundamental study in the field of colliding gravitational waves is the KP solution which describes the collision of two impulsive gravitational waves with parallel polarization. The generalization of KP solution for nonaligned polarization is given by the NH solution. The latter is rederived by CF [12] by employing the Ernst formalism and it is shown that, if the solution to Eq.(59) is taken as

$$E = p\eta + iq\mu, \quad \text{with} \quad p^2 + q^2 = 1, \quad (70)$$

solution to the metric functions leads to the NH solution. Note that previous choice with $q \rightarrow -q$ for the Ernst equation is admissible provided we employ the same convention throughout. Note that q measures the second (or cross) polarization of the waves involved. With $q = 0$ ($p = 1$) this solution reduces to the solution of linearly polarized wave problem of KP. It is important to note that, the NH solution is not of Petrov - type D. However, if the same function in Eq.(70) is considered for E^\dagger for the Ernst equation (66), i.e.

$$E^\dagger = p\eta + iq\mu, \quad \text{with} \quad p^2 + q^2 = 1, \quad (71)$$

yields the conjugate expression

$$Z^\dagger = \Psi + i\Phi = \frac{1 + p\eta + iq\mu}{1 - p\eta - iq\mu}, \quad (72)$$

from which we can find Ψ and Φ separately as,

$$\Psi = \frac{1 - p^2\eta^2 - q^2\mu^2}{(1 - p\eta)^2 + q^2\mu^2}, \quad \text{and} \quad \Phi = \frac{2q\mu}{(1 - p\eta)^2 + q^2\mu^2}. \quad (73)$$

The metric function χ is readily available from Eq.(64) as,

$$\chi = \sqrt{\Delta\delta} \frac{(1 - p\eta)^2 + q^2\mu^2}{1 - p^2\eta^2 - q^2\mu^2}, \quad (74)$$

while q_2 is obtained from Eq.(62) by simple integration as,

$$q_2 = \frac{2q}{p(1 + p)} - \frac{2q\delta(1 - p\eta)}{p(1 - p^2\eta^2 - q^2\mu^2)}. \quad (75)$$

The remaining metric function $\nu + \mu_3$ can be obtained by using equations (68) and (69) and the result is

$$e^{\nu + \mu_3} = \frac{(1 - p\eta)^2 + q^2\mu^2}{\sqrt{\Delta}}. \quad (76)$$

Consequently, the metric that describes the collision of gravitational waves for the assumed solution of Ernst equation is

$$ds^2 = X \left[\frac{(d\eta)^2}{\Delta} - \frac{(d\mu)^2}{\delta} \right] - \Delta\delta \frac{X}{Y} dy^2 - \frac{Y}{X} (dx - q_2 dy)^2, \quad (77)$$

where

$$X = (1 - p\eta)^2 + q^2\mu^2, \quad Y = 1 - |(E^\dagger)|^2 = 1 - p^2\eta^2 - q^2\mu^2 = p^2\Delta + q^2\delta, \quad (78)$$

and

$$q_2 = \frac{2q}{p(1+p)} - \frac{2q\delta(1-p\eta)}{pY}, \quad (79)$$

CX proceeded also with the analytic extension of their solution of colliding waves. In this formalism different universes are patched together, ending up with new universes as white holes. Example of more general analytic extension with more parameters such as charge [35] and Newman-Unti-Tamburino (NUT) [36] also is available in the literature [37]

2. Extension of the Interaction Region into Incoming Regions

In order to find the wave profiles that participate in the collision, the metric obtained in the interaction region should be extended to the incoming regions. The approaching wave in one of the plane symmetric Region II ($u \geq 0, v < 0$) is obtained by dropping the v in the metric (77). This process follows systematically with $v < 0$, since the step function $\theta(v) = 0$ for $v < 0$ and the v dependence disappears. This is achieved by the substitution $\eta = \mu = \sin(u\theta(u))$, so that the metric functions take the form

$$X(u) = 1 - 2p \sin u + \sin^2 u, \quad Y(u) = \Delta = \delta = \cos^2 u, \quad (80)$$

$$q_2(u) = \frac{2q}{(1+p)} [(1+p) \sin u - 1]. \quad (81)$$

in which as described above u is implied with the step function. Hence, the metric in Region II in terms of the null coordinate u can be expressed as

$$ds^2 = \frac{2X(u)}{\sqrt{1-u^2}} dudv - (1-u^2) \left[X(u) dy^2 + \frac{1}{X(u)} (d\tilde{x} - 2q \sin u dy)^2 \right], \quad (82)$$

where

$$\tilde{x} = x + \frac{2q}{(1+p)} y. \quad (83)$$

The plane symmetric metric (82) has a single curvature tensor component which describes the profile of the incoming gravitational wave given by

$$\Psi_4 = -(p - iq) \delta(u) - \frac{3(X - 2iq \sin u)}{X^4 \sqrt{X^2 + 4q^2 \sin^2 u}} \frac{(1 - p \sin u - iq \sin u)^3}{(p + iq)^2} \theta(u), \quad (84)$$

in which $\delta(u)$ stands for the Dirac delta function and $X = X(u)$ is given (80). Consequently, the incoming wave is a composition of an impulsive and shock gravitational waves. Similar incoming wave profile $\Psi_0(v)$ from the Region III ($u < 0, v \geq 0$) is obtained by the substitution $\eta = -\mu = \sin(v\theta(v))$, which will not be given.

3. CX - Duality leading to the Kerr Metric

The Petrov classification of the metric (77), as is shown by CX is type - D. Calculations for the Weyl scalar with a proper tetrads reveals the only nonvanishing scalar as

$$\Psi_2 = \frac{1}{2(1 - p\eta - iq\mu)^3}. \quad (85)$$

The Weyl scalar Ψ_2 , in the terminology of the CGW is interpreted as the Coulomb component and arises as a result of the non-linear interaction. The unboundedness of Ψ_2 indicates the existence of the curvature singularity. As it was shown in the KP and NH solutions, there is a curvature singularity in the region of interaction when $u^2 + v^2 = 1$. This surface corresponds to $\eta = 1$, for the metric (77). And hence, the behaviour of Ψ_2 is finite which indicates Killing - Cauchy horizon instead of a curvature singularity.

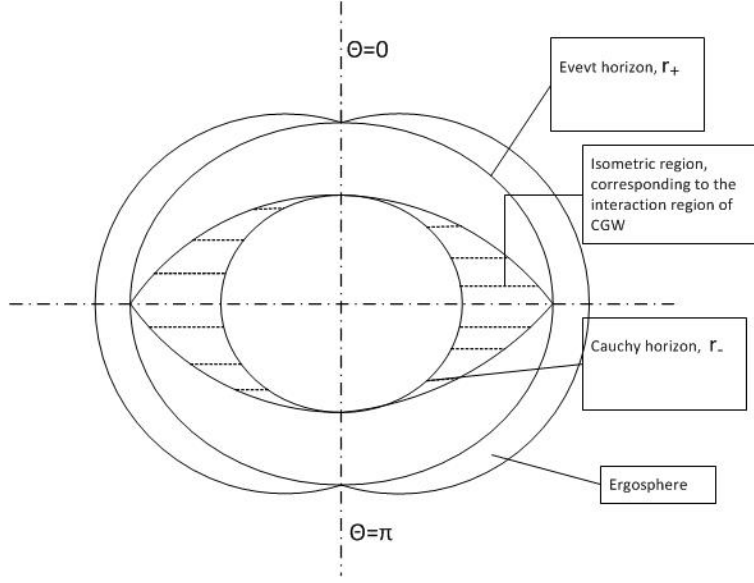


FIG. 2: The figure represent a partial correspondence between the Kerr black hole interior and the interaction region of colliding plane waves. The shaded region correspond to the interaction region of colliding gravitational plane wave solution (Eq. 72) which is locally isometric to the part of the region in between the Cauchy (r_-) and event (r_+) horizons of the Kerr black hole. The location of the Cauchy horizon of colliding plane wave $\eta = 1$, corresponds to the inner horizon (r_-) of the Kerr black hole located at $r_- = M - \sqrt{M^2 - a^2}$. The collision occurs at $\eta = \mu = 0$, which is equivalent to (from Eq. 81) $r = M$ and $\theta = \pi/2$, in the corresponding Kerr geometry.

We apply the following transformations to the metric (77) which describes the interaction region of the collision of impulsive and shock gravitational waves,

$$t = M \left(x - \frac{2q}{p(1+p)}y \right), \quad \phi = \frac{M}{\sqrt{M^2 - a^2}}y, \quad \eta = \pm \frac{(M - r)}{\sqrt{M^2 - a^2}}, \quad \mu = \cos \theta, \quad (86)$$

with

$$p = \pm \frac{\sqrt{M^2 - a^2}}{M}, \quad q = \pm \frac{a}{M}, \quad \text{and} \quad M^2 > a^2. \quad (87)$$

such that $p^2 + q^2 = 1$. We have the correspondence [11], accordingly

$$1 - p\eta = \frac{r}{M}, \quad 1 - \eta^2 = -\frac{\tilde{\Delta}}{M^2 - a^2}, \quad (88)$$

in which $\tilde{\Delta}$ stands for the horizon function

$$\tilde{\Delta} = r^2 - 2Mr + a^2 = (r - r_-)(r - r_+). \quad (89)$$

We note that this transformation (86-87) is suggestive upon the striking similarity between (72) and (25) in which the Killing fields ξ_x and ξ_y in one are replaced by the ξ_t and ξ_φ in the other. An interesting aspect of this transformation is that the infinite ranged coordinate y is mapped into the compact coordinate φ . The non-orthogonality of Killing vectors in one is provided by q (the relative polarization of waves) in one while by a (rotation) in the other. It is natural that in the transformation one (q) will be related to the other (a). These substitutions transform the metric(77) to the following form,

$$M^2 ds^2 = \left(\frac{\tilde{\Delta} - a^2 \sin^2 \theta}{\rho^2} \right) \left[dt + \frac{2aMr \sin^2 \theta}{\tilde{\Delta} - a^2 \sin^2 \theta} d\phi \right]^2 - \frac{\rho^2}{\tilde{\Delta}} \left[dr^2 + \tilde{\Delta} d\theta^2 \right] - \left[\frac{\tilde{\Delta} \rho^2 \sin^2 \theta}{\tilde{\Delta} - a^2 \sin^2 \theta} \right] d\phi^2, \quad (90)$$

where $\rho^2 = r^2 + a^2 \cos^2 \theta$, and the constants M and a represents the emergent parameters in the local isometry for mass and rotation, respectively. The roots of $\tilde{\Delta}$, namely r_+ and r_- are the event (outer) and Cauchy (inner) horizons,

respectively. From this transformation, we conclude that the region of interaction is locally isometric to the region in between the inner and outer horizon of the Kerr black hole. The corresponding Weyl scalar in Boyer - Lindquist coordinates becomes,

$$\Psi_2 = -\frac{M}{(r - ia \cos \theta)^3}. \quad (91)$$

Figure 2, illustrates the region which is identical both in Kerr black hole and in the interaction region of the CGW.

One may naturally look for a corresponding isometric colliding wave solution belonging to the overspinning case $a > M$. This solution has not been studied, however, the study which may be adopted to achieve such a solution has been considered in [43]. In this paper, the prolate coordinate system is used to obtain vacuum cylindrical gravitational waves. In obtaining the solution, the Ernst formalism was employed. The solution to the Ernst equation is obtained for ξ^{-1} , instead of ξ ($= p\eta + iq\mu$) and the condition $q^2 - p^2 = 1$, instead of $p^2 + q^2 = 1$. With this formalism it is shown in [43] that, the overspinning Kerr solution can be derived via complex transformation from a class of cylindrical wave spacetime. The scope of [43] was not to derive a corresponding colliding gravitational wave solution but to connect overspinning Kerr with a cylindrical wave spacetime. The fact that $q^2 - p^2 = 1$ yields hyperbolic functions for η and μ is inappropriate for the colliding wave problem. Assuming that such functions are expressed in the null coordinates u and v , the boundary conditions for the incoming regions will not be satisfied across the null boundaries. This naturally give rise to Dirac delta function sources at the boundaries which is not acceptable for a vacuum problem. In short, correspondence of overspinning Kerr solution to a CGW problem remains open.

IV. QUANTUM PROBES OF TIMELIKE KERR NAKED SINGULARITY

In this section we seek to answer the following question: Is the only classical singularity of Kerr, namely, $r = 0$ and $\theta = \pi/2$, also quantum singular ?.

In Boyer - Lindquist coordinates (t, r, θ, ϕ) , the Kerr metric can be written as,

$$ds^2 = -\frac{\tilde{\Delta}}{\rho^2} [dt - a \sin^2 \theta d\phi]^2 + \frac{\rho^2}{\tilde{\Delta}} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2 \theta}{\rho^2} [adt - (r^2 - a^2) d\phi]^2, \quad (92)$$

note that the signature of the metric is changed to $+2$. If the rotational parameter dominates the mass parameter (over spinning case, $a > M$), there are no horizons and the timelike naked singularity at $r = 0$ and $\theta = \pi/2$ is developed for asymptotic observers. The metric (92) for the particular case of $M = 1$ is reduced to the following form, in the equatorial plane $\theta = \pi/2$,

$$ds^2 = -\left(1 - \frac{2}{r}\right) dt^2 - \frac{4a}{r} dt d\phi + \frac{r^2}{\tilde{\Delta}} dr^2 + r^2 d\theta^2 + \left(r^2 + a^2 + \frac{2a^2}{r}\right) d\phi^2, \quad (93)$$

in which $\tilde{\Delta} = r^2 + a^2 - 2r$, and the ranges of the coordinates vary as

$$0 \leq r \leq \infty, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi < 2\pi. \quad (94)$$

In this case, the topology of the Kerr metric changes from ergospheres to ergo torus in which the inner circle is the ring singularity and has a timelike character (see [25] for figures). The timelike Kerr naked singularity can be explained much better if we switched the coordinates to the so-called Kerr - Schild [39] form in which the structure of $r = 0$ surface becomes more transparent. The so-called Kerr - Schild forms (t, x, y, z) are defined by

$$x = \sqrt{r^2 + a^2} \sin \theta \cos \left(\phi + \arctan \frac{a}{r} \right), \quad (95)$$

$$y = \sqrt{r^2 + a^2} \sin \theta \sin \left(\phi + \arctan \frac{a}{r} \right), \quad (96)$$

$$z = r \cos \theta. \quad (97)$$

We have

$$x^2 + y^2 + z^2 = (r^2 + a^2) \sin^2 \theta + r^2 \cos^2 \theta. \quad (98)$$

Since the Weyl scalar (91) diverges at $r = 0$ and $\theta = \pi/2$, this implies

$$x^2 + y^2 = a^2 \quad \text{at} \quad z = 0. \quad (99)$$

This is in fact a ring that forms the boundary of a disc. Hence, the only singularity of the Kerr spacetime is located along this ring and the interior of the ring $x^2 + y^2 < a^2$ remains regular.

In classical general relativity, the formation of a naked singularity can be attributed as a threat to the cosmic censorship hypothesis of Penrose, because, all singularities of gravitational collapse must be hidden within black holes. Hence, the resolution of naked singularities constitute one of the unresolved problem of black hole physics. It is believed that, the well established quantum theory of gravity will be a powerful tool for the resolution, however, the complete quantum theory of gravity is still under "construction". In the literature, there are alternative methods for this purpose. Loop quantum gravity [26] and string theory [27, 28] are the two important study field in the resolution of singularities.

In this section, another alternative method will be used. The formation of timelike naked singularities in the fast rotating case will be investigated in view of quantum mechanics. Instead of point particle probes which leads to the notion of *geodesics incompleteness*, the wave probe will be used which leads to the notion of *quantum singularity*. In doing this, the work of Wald [29] which was developed by Horowitz and Marolf (HM) [30] for static spacetimes is extended to stationary metrics. However, this extension is not a general extension. The main theme of the HM criterion is to split the spatial and time part of the Klein- Gordon equation and write it in the form of

$$\frac{\partial^2 \psi}{\partial t^2} = -A\psi, \quad (100)$$

where A is the spatial wave operator. Note that, this operator is a symmetric and positive operator on the Hilbert space \mathcal{H} . According to the HM, the singular character of the spacetime with respect to wave probe is characterized by investigating whether the spatial part of the wave operator A has a unique self - adjoint extensions (i.e. essentially self - adjoint) in the entire space or not. If the extension is unique, it is said that the space is quantum mechanically regular. In order to make this point more clear, consider the Klein- Gordon equation for a free particle that satisfies

$$i \frac{d\psi}{dt} = \sqrt{A_E} \psi, \quad (101)$$

whose solution is

$$\psi(t) = e^{-it\sqrt{A_E}} \psi(0), \quad (102)$$

in which A_E denotes the extension of the operator A . If A has not a unique self - adjoint extensions, then the future time evolution of the wave function (102) is ambiguous. And, HM criterion defines the spacetime as quantum mechanically singular (see [31], for a detailed mathematical background).

The timelike naked singularity for the Kerr metric will be probed with scalar waves satisfying the Klein-Gordon equation

$$\left(\frac{1}{\sqrt{g}} \partial_\mu [\sqrt{g} g^{\mu\nu} \partial_\nu] - \tilde{m}^2 \right) \psi = 0, \quad (103)$$

in which \tilde{m} is the mass of the scalar particle. The considered model of solution to the equation (103) is called a reduced wave equation which admits solution in the form:

$$\psi(t, r, \theta, \phi) = e^{i(k\phi)} f(t, r, \theta). \quad (104)$$

This form of choice for the solution of the Klein-Gordon equation is also considered in [32], for exploring the new symmetries of the solution of the wave equation. For the metric (93), the Klein- Gordon equation with the assumed solution can be written as

$$\frac{\partial^2 f}{\partial t^2} + \frac{4aki}{r\Xi} \frac{\partial f}{\partial t} = \frac{\bar{\Delta}}{r^2\Xi} \left\{ \frac{\partial}{\partial r} \left(\bar{\Delta} \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial \theta^2} - \left[k^2 \left(1 - \frac{a^2}{\bar{\Delta}} \right) + \tilde{m}^2 r^2 \right] \right\} f, \quad (105)$$

in which $f = f(t, r, \theta)$ and k can take values of all integers which is associated with the orbital quantum number and $\Xi = r^2 + a^2 + \frac{2a^2}{r}$. The present form of equation (105), is not suitable to use because the temporal and spatial parts are not yet seperable. The second term on the left hand side of the equation (105) arose due to the presence of $g_{t\phi}$ term in the metric (93). However, it is crucial to know that the formed timelike naked singularity of the Kerr metric

has some interesting properties that is not shared by the any other naked singularities forming in static spacetimes. The Kerr naked singularity becomes visible to asymptotic observers, if one approaches to the singularity $r = 0$ from $\theta = \pi/2$ only. In other words, the surface $r = 0$, is a disc with a boundary of a ring singularity. The trajectories that approach to this surface $r = 0$ with $\theta \neq \pi/2$, do not fall into the singularity, and hence, all points are regular.

In the assumed solution in equation (104), the constant parameter k which runs for all integer values is related to the orbital quantum number corresponding to the projection of the angular momentum onto the axis of symmetry. In order to probe the Kerr naked singularity with waves, the wave should propagate along the equilateral plane $\theta = \pi/2$. In addition to this, the ring singularity is located at $r = 0$ surface. These restrictions on the wave propagation imposes the condition that the only wave mode available for this probe is the *s-wave* mode that corresponds to $k = 0$. This coincidence enables the equation (105) separable in time and spatial part as

$$\frac{\partial^2 f}{\partial t^2} = \frac{\bar{\Delta}}{r^2 \Xi} \left\{ \frac{\partial}{\partial r} \left(\bar{\Delta} \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial \theta^2} - \tilde{m}^2 r^2 \right\} f, \quad (106)$$

and the spatial wave operator A which will be investigated for a unique self - adjoint extension has a form of

$$A = -\frac{\bar{\Delta}}{r^2 \Xi} \left\{ \frac{\partial}{\partial r} \left(\bar{\Delta} \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial \theta^2} - \tilde{m}^2 r^2 \right\}. \quad (107)$$

The problem now is to count the number of extensions of the operator A . This is done by using the concept of deficiency indices discovered by Weyl [33] and generalized by von Neumann [34] (see [31] for a detailed mathematical background). The determination of the deficiency indices (n_+, n_-) of the operator A , is reduced to count the number of solutions to equation

$$A\psi \pm i\psi = 0, \quad (108)$$

that belong to the Hilbert space \mathcal{H} . If there are no square integrable ($L^2(0, \infty)$) solutions (i.e., $n_+ = n_- = 0$) in the entire space, the operator A possesses a unique self-adjoint extension and it is called essentially self-adjoint. Consequently, the method to find a sufficient condition for the operator A to be essentially self-adjoint is to investigate the solutions satisfying equation (108) that do not belong to the Hilbert space \mathcal{H} .

The solution to Eq.(108) is obtained by assuming the solution in separable form $\psi = R(r)Y(\theta)$, which yields the radial equation as

$$R'' + \frac{2(r-1)}{\bar{\Delta}} R' - \frac{1}{\bar{\Delta}} \left[r^2 \left(\tilde{m}^2 \pm \frac{i\Xi}{\bar{\Delta}} \right) + c \right] R = 0, \quad (109)$$

in which prime denotes the derivative with respect to r and c is a real separation constant.

The square integrability of the solutions of Eq.(109) for each sign \pm is checked by calculating the squared norm of the solution of Eq.(109) for the massless wave $\tilde{m} = 0$, in which the function space on each $t = \text{constant}$ hypersurface Σ_t is defined as $\mathcal{H} = \{R \mid \|R\| < \infty\}$. The squared norm can be defined as [30],

$$\|R\|^2 = \int_{\Sigma_t} \sqrt{-g} g^{tt} R R^* d^3 \Sigma_t. \quad (110)$$

The spatial operator A is essentially self-adjoint if neither of the solutions of Eq.(109) is square integrable over all space $L^2(0, \infty)$. The behavior of the Eq.(109) near $r \rightarrow 0$ and $r \rightarrow \infty$ will be considered separately in the following subsections.

A. The case of $r \rightarrow 0$:

In the case when $r \rightarrow 0$, the Eq.(109) simplifies to,

$$\psi'' - \frac{2}{a^2} \psi' - \frac{c}{a^2} \psi = 0. \quad (111)$$

If the separation constant $c > -\frac{1}{a^2}$, then the solution is

$$R(r) = e^{r/a^2} \left(C_1 e^{\alpha_1 r/a^2} + C_2 e^{-\alpha_1 r/a^2} \right), \quad (112)$$

in which $\alpha_1 = \sqrt{1 + ca^2}$. If the separation constant $c < -\frac{1}{a^2}$, then $\alpha_1 \rightarrow i\beta_1$, where β_1 is an arbitrary real constant and the solution is

$$R(r) = e^{r/a^2} \left(C_3 e^{i\beta_1 r/a^2} + C_4 e^{-i\beta_1 r/a^2} \right), \quad (113)$$

in which C_i ($i = 1, 2, 3, 4$) are the integration constants.

The square integrability of the solution (112) and (113) are checked by calculating the squared norm defined in equation (110) in the limiting case of the metric (93) when $r \rightarrow 0$, which is given by

$$\|R\|^2 \sim \int_0^{const.} r^{5/2} |R|^2 dr. \quad (114)$$

The analysis has revealed that, if the separation constant $c < -\frac{1}{a^2}$, the solution (113) is square integrable, since $\|R\|^2 < \infty$, thus, the solution belongs to the Hilbert space. But, there is a specific case for $c > -\frac{1}{a^2}$ in the solution (112) such that, if the separation constant is chosen very large, then, this specific solution fails to satisfy square integrability condition, i.e. $\|R\|^2 \rightarrow \infty$.

B. The case of $r \rightarrow \infty$:

When $r \rightarrow \infty$, the Eq.(109) reduces to

$$R'' + \frac{2}{r}R' \pm iR = 0, \quad (115)$$

whose solution is given by

$$R(r) = \frac{1}{r} (C_3 \sin(\alpha_2 r) + C_4 \cos(\alpha_2 r)), \quad (116)$$

in which $\alpha_2 = \frac{1}{2}\sqrt{\pm 1 + i}$, and C_3 and C_4 are the integration constants. The square integrability is checked with the following norm written for the case $r \rightarrow \infty$,

$$\|R\|^2 \sim \int_{const.}^{\infty} r^2 |R|^2 dr. \quad (117)$$

The result is that the solution fails to satisfy square integrability condition ($\|R\|^2 \rightarrow \infty$), and hence, does not belong to the Hilbert space.

The method of defining whether the operator A has a unique self-adjoint extension (or essentially self-adjoint) is to investigate the solution of Eq.(109) in the entire space $(0, \infty)$ and count the number of solutions that do not belong to the Hilbert space. In other words, if there is one solution that fails to be square integrable for the entire space then the operator A is said to be essentially self-adjoint. Our analysis has shown that the behaviour of Eq. (109), when $r \rightarrow \infty$, admits solution that is not square integrable. Hence, the operator A is essentially self-adjoint and the future time evolution of the quantum particles/waves can be predicted uniquely. Consequently, the classical Kerr naked ring singularity is healed and becomes quantum mechanically regular when probed with particles/waves described by the Klein-Gordon equation.

We would like to emphasize that the criterion proposed by HM [30], by adopting the earlier work of Wald [29], is used for probing the timelike singularities in static spacetimes. A quantum mechanical particle with mass \tilde{m} can be described by the Klein - Gordon equation in the form of Eq. (100). In this equation, there is a complete separation in the time and spatial parts. Hence, the operator A is completely defined in terms of spatial coordinates and spatial derivatives only. This leads to Eq.(102), that may be interpreted as, translating the well-posedness of the initial value problem into the essential self-adjointness of the operator A [31]. In this paper, the addressed problem is the Kerr timelike naked ring singularity that developed in stationary spacetime. The Klein-Gordon equation for this spacetime is given in Eq. (105). Since, complete separation in the spatial and time derivatives is not possible, one may define the right hand side of Eq. (105) as the spatial part of the reduced normalized wave operator [32], without imposing $k = 0$ (*s-wave*). However, even if we consider this case, the result would not change, because, the constant number k do not contribute near $r \rightarrow 0$ and $r \rightarrow \infty$. As a result, we would obtain exactly the same behaviour of the operator A given in Eq.(106).

V. CONCLUSION AND DISCUSSION

The local isometry between a Kerr black hole and a CGW spacetime is known as the CX duality. A CH forming CGW spacetime transforms locally by a coordinate transformation into a black hole metric. Is this a coincidence ?. Can such a duality be valid for all black holes / colliding shock waves ?. A stable black hole from classical physics point of view is a highly localized, simplest object in our universe. It is a long time, more than a half century that stability of black holes is investigated. From quantum physics point of view it was shown by Hawking in 1970's that black holes undergo a thermal radiation. Such a radiation causes the black hole to evaporate completely or leave behind a stable remnant. Let us remember also that in a quantum theory, due to the uncertainty principle even exact location of the horizon is questionable: a fuzzy picture becomes indispensable at the Planck scale. Different classes of CGW such as KP and NH form naked singularities instead of the analytically extendible "beautiful" CH, so that an observer falls into the singularity without crossing an event horizon. Further, the beauty of CH in CGW is destroyed by the ugly fact: it is not stable against various perturbations so that it settles down to a naked singularity. A naked singularity is known to violate the cosmic censorship conjecture. For a diagonal metric such as Schwarzschild, once the observer crosses the event horizon she can not distinguish between a black hole and colliding wave spacetimes. (Note that for an off - diagonal black hole metric such as Kerr, there is an extra region, the ergosphere, depicted in Fig. (2)). This is a lesson that we learn from the CX duality. It is our belief that the key to understand quantum gravity through the wave - particle duality lies in understanding the event horizon. Direct experimental discovery of an event horizon either in space or in analog gravity models in a laboratory will highlight a "miracle without miracle" in John A. Wheeler's terminology. We may anticipate, without a priori proof, that solutions to information paradox, entanglement, complementarity, firewall and other mind boggling concepts can be tackled all with a thorough understanding of the physics of horizons [38]. Let the Kerr geometry inspire / guide us more and more toward this goal.

Finally, the formation of the timelike Kerr naked singularity in the overspinning case is analyzed in view of quantum mechanics with the criterion proposed by HM. This singularity is probed with waves obeying the Klein-Gordon equation. Analysis has revealed that, in order for probing the ring singularity, the spatial derivative operator of the reduced Klein-Gordon equation is defined for a specific wave mode, namely the *s-wave*. It is shown that the spatial derivative operator A defined in Eq. (106) is essentially self-adjoint. As a result, the classical Kerr ring singularity is quantum regular with respect to the wave probe described by the reduced Klein-Gordon equation.

The wave mode used in this study is the only wave mode that makes it possible to use the criterion proposed by HM. The general extension of the criterion of HM for stationary spacetimes has not been achieved yet and this problem is an open problem. Although, the preliminary work for stationary spacetimes is considered in [44], however, the formulation has not been fully completed.

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